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An Abstract Nonlinear Stochastic Integral Equation

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1. INTRODUCTION

Consider the nonlinear stochastic integral equation

$$X_z = \varphi_z + \int_0^z K_1(u, X)\, dM_u + \int_0^z L_1\left(u, X, \int_0^{\cdot} W_1(u, v, X)\, dM_v\right) dM_u$$
$$+ \int_0^z K_2(u, X)\, du + \int_0^z L_2\left(u, X, \int_0^{\cdot} W_2(u, v, X)\, dv\right) du \qquad (1.1)$$

in an abstract Wiener space (i, H, B), where (M_u) is a B-valued d-parameter Wiener process. The first three integrals in (1.1) are in the sense of Ito and the last in the sense of Bochner.

The linear case ($L_1 = L_2 = 0$) was considered by many authors, for example, by Ito [7], McKean [12], Gihman and Skorohod [5], Kazamaki [8], Doléans-Dade [2], and Protter [14] when $d = 1$ and the state space is finite dimensional, by Gihman [4], Tudor [17], and Yeh [19] when $d = 2$ and the state space is finite dimensional, by Kuo [9, 10], Skorohod [15, 16], and Metivier and Pellaumail [13] when $d = 1$ and the state space is infinite dimensional and by Tudor [18] when $d = 2$ and the state space is infinite dimensional.

Our purpose is to give conditions which guarantee existence, uniqueness and continuous dependence of solutions to (1.1).

2. PRELIMINARIES

Let H be a real separable Hilbert space with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$. $\mathscr{L}_2(H)$ denotes the Hilbert space of all Hilbert–Schmidt operators on H with norm $\|\cdot\|_2$.

If $\|\cdot\|$ is a measurable norm on H then B denotes the completion of H with respect to this norm. The triple (i, H, B), where $i: H \to B$ is the injection, is called an abstract Wiener space.

Note that there exists a constant c such that $\|x\| \leqslant c\,|x|$ for each $x \in H$ [10, Lemma 4.2, p. 56].

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For an integer $d \geqslant 1$ we write: $\Omega_d^H = C(R_+^d, H)$ (resp. $\Omega_d^B = C(R_+^d, B)$) for the collection of all continuous functions from R_+^d to H (resp. from R_+^d to B) considered as complete separable metric spaces with respect to the uniform convergence on compact subsets, and $\mathscr{F}_z^H$ (resp. $\mathscr{F}_z^B$) for the Borel field generated by x_u^H, $u \leqslant z$ (resp. x_u^B; $u \leqslant z$), where x_z^H and x_z^B are canonical processes on Ω_d^H, Ω_d^B, respectively.

If D is a rectangle in R_+^d then we put $\lambda(D)$ for its Lebesgue measure. We consider (M_u) a B-valued d-parameter Wiener process relative to a filtration $(\mathscr{F}_u)$. Recall that (M_u) is a $\mathscr{F}_u$-Markov process with R_+^d as time parameter, with B as state space such that it vanishes on the axes, and has continuous sample paths and $P_u(x, A) = p_{\lambda([0,u])}(A - x)$ as transition function (here p_t is the Wiener measure on B of parameter t). A construction of (M_u) can be found in Gross [6] for $d = 1$. The construction for $d \geqslant 2$ goes in a similar manner.

If $K: R_+^d \times \Omega_d^B \to S$, $L: R_+^d \times \Omega_d^B \times \Omega_d^H \to T$, $W: R_+^{2d} \times \Omega_d^B \to U$ (S, T, U are Banach spaces with norms $\|\cdot\|_S$, $\|\cdot\|_T$, $\|\cdot\|_U$) then we make the following assumptions:

(A) There exists a Radon measure α on R_+^d such that for every z there is a constant $c(z)$ with properties

$$\|K(u,f) - K(u,g)\|_S^2 + \|W(v,u,f) - W(v,u,g)\|_U^2$$
$$\leqslant c(z)\left\{\|f(u) - g(u)\|^2 + \int_0^u \|f(u_1) - g(u_1)\|^2 \, d\alpha(u_1)\right\}$$

$$\|L(u,f,f_1) - L(u,g,g_1)\|_T^2$$
$$\leqslant c(z)\left\{\|f(u) - g(u)\|^2 + |f_1(u) - g_1(u)|^2 \right.$$
$$\left. + \int_0^u [\|f(u_1) - g(u_1)\|^2 + |f_1(u_1) - g_1(u_1)|^2] \, d\alpha(u_1)\right\}$$

for each $u \leqslant z$, $v \in R_+^d$, $f, g \in \Omega_d^B$, $f_1, g_1 \in \Omega_d^H$.

(B) There exists a Radon measure β on R_+^d such that for every z there is a constant $d(z)$ with properties

$$\|W(v,u,f)\|_U^2 \leqslant d(z)\left\{1 + \|f(u)\|^2 + \int_0^u \|f(u_1)\|^2 \, d\beta(u_1)\right\}$$

$$\|L(u,f,g)\|_T^2 \leqslant d(z)\left\{1 + \|f(u)\|^2 + |g(u)|^2 \right.$$
$$\left. + \int_0^u [\|f(u_1)\|^2 + |g(u_1)|^2] \, d\beta(u_1)\right\}$$

for each $u \leqslant z$, $v \in R_+^d$, $f \in \Omega_d^B$, $g \in \Omega_d^H$.

3. Uniqueness Theorems

Suppose we are given the continuous B-valued process $\varphi(u)$ and

$$K_1: R_+^d \times \Omega_d^B \to \mathscr{L}_2(H); \qquad K_2: R_+^d \times \Omega_d^B \to B$$
$$L_1: R_+^d \times \Omega_d^B \times \Omega_d^H \to \mathscr{L}_2(H); \qquad L_2: R_+^d \times \Omega_d^B \times \Omega_d^B \to B$$
$$W_1: R_+^{2d} \times \Omega_d^B \to \mathscr{L}_2(H); \qquad W_2: R_+^{2d} \times \Omega_d^B \to B$$

such that

(i_1) K_i, L_i, W_i are measurable in all arguments,

(i_2) for every $u, v, K_i(u, \cdot)$ are $\mathscr{F}_u^B$-measurable, $L_i(u, \cdot)$ are $\mathscr{F}_u^B \otimes \mathscr{F}_u^H$-measurable, $W_i(u, v, \cdot)$ are $\mathscr{F}_v^B$-measurable.

THEOREM 3.1. *If K_i, L_i, W_i defined as above satisfy assumption* (A) *then the solution of* (1.1) *is pathwise unique.*

Proof. Let X, Y be two solutions of (1.1). Define the process

$$\varphi_z^N = \chi_{\{\sup_{u \leqslant z}(\|X_u\|^2 + \|Y_u\|^2) \leqslant N\}}.$$

We have

$$\varphi_z^N \|X_z - Y_z\|^2 \leqslant 4\varphi_z^N \sum_{i=1}^{4} J_i(z) \tag{3.1}$$

where

$$J_1(z) = \left\| \int_0^z \varphi_u^N [K_1(u, X) - K_1(u, Y)]\, dM_u \right\|^2$$

$$J_2(z) = \left\| \int_0^z \varphi_u^N \left[L_1\left(u, X, \int_0^{\cdot} W_1(u, v, X)\, dM_v\right) \right.\right.$$
$$\left.\left. - L_1\left(u, Y, \int_0^{\cdot} W_1(u, v, Y)\, dM_v\right) \right] dM_u \right\|^2$$

$$J_3(z) = \left\| \int_0^z \varphi_u^N [K_2(u, X) - K_2(u, Y)]\, du \right\|^2$$

$$J_4(z) = \left\| \int_0^z \varphi_u^N \left[L_2\left(u, X, \int_0^{\cdot} W_2(u, v, X)\, dv\right) \right.\right.$$
$$\left.\left. - L_2\left(u, Y, \int_0^{\cdot} W_2(u, v, Y)\, dv\right) \right] du \right\|^2.$$

If we put $F(u)=E(\varphi_u^N \|X_u - Y_u\|^2)$ then using the hypotheses, Doob's inequality and the Cauchy–Schwartz inequality we obtain for $z \leqslant z_0$

$$E[J_1(z)] \leqslant c(z_0)\, c \left\{ \int_0^z F(u)\, du + \int_0^z \int_0^u F(v)\, d\alpha(v)\, du \right\}$$

$$\begin{aligned} E[J_2(z)] \leqslant {} & c(z_0)\, c \left\{ \int_0^z F(u)\, du + \int_0^z \int_0^u F(v)\, d\alpha(v)\, du \right\} \\ & + c^2(z_0)\, c \left\{ \int_0^z \int_0^u F(v)\, dv\, du + \int_0^z \int_0^u \int_0^v F(w)\, d\alpha(w)\, dv\, du \right. \\ & + \int_0^z \int_0^u \int_0^v F(w)\, dw\, d\alpha(v)\, du \\ & \left. + \int_0^z \int_0^u \int_0^v \int_0^w F(p)\, d\alpha(p)\, dw\, d\alpha(v)\, du \right\} \end{aligned}$$

$$E[J_3(z)] \leqslant c(z_0)\, \lambda([0, z_0]) \left\{ \int_0^z F(u)\, du + \int_0^z \int_0^u F(v)\, d\alpha(v)\, du \right\}$$

$$\begin{aligned} E[J_4(z)] \leqslant {} & c(z_0)\, \lambda([0, z_0]) \left\{ \int_0^z F(u)\, du + \int_0^z \int_0^u F(v)\, d\alpha(v)\, du \right\} \\ & + c^2(z_0)\, \lambda^2([0, z_0]) \left\{ \int_0^z \int_0^u F(v)\, dv\, du + \int_0^z \int_0^u \int_0^v F(w)\, dw\, d\alpha(v)\, du \right. \\ & \left. + \int_0^z \int_0^u \int_0^v \int_0^w F(p)\, d\alpha(p)\, dw\, d\alpha(v)\, du \right\}. \end{aligned}$$

Taking the expectation in (3.1) we get

$$\begin{aligned} F(z) \leqslant c_1(z_0) & \left\{ \int_0^z F(u)\, du + \int_0^z \int_0^u F(v)\, dv\, du + \int_0^z \int_0^u F(v)\, d\alpha(v)\, du \right. \\ & \left. + \int_0^z \int_0^u \int_0^v F(w)\, dw\, d\alpha(v)\, du + \int_0^z \int_0^u \int_0^v \int_0^w F(p)\, d\alpha(p)\, dw\, d\alpha(v)\, du \right\}. \end{aligned} \tag{3.2}$$

If we denote $G(z) = \sup_{u \leqslant z} F(u)$ then (3.2) implies

$$F(z) \leqslant c_2(z_0) \int_0^z G(u)\, du$$

whence

$$G(z) \leqslant c_2(z_0) \int_0^z G(u)\, du$$

so we may apply the Bellman–Gronwall inequality to conclude that $G(z) = 0$ for $z \leqslant z_0$. Therefore for each z we have $X_z = Y_z$ a.s. and since X and Y are continuous it follows that X and Y are indistinguishable. This completes the proof of theorem.

Remark. The previous theorem is an extension to the case of infinite-dimensional state space of the Lipter and Siriaev result [11, Theorem 4.6, p. 148] and of the Yeh result [19, Theorem 3.8, p. 228].

THEOREM 3.2 (Ito's Formula). *Let $f: R_+ \times B^2 \to R$ be a continuous function. Suppose that for every $x, y \in B$, $f(\cdot, x, y)$ has the derivative f_t continuous on $R_+ \times B^2$ and that for every $t \geqslant 0$, $f(t, \cdot)$ has the partial H-derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ which are continuous on $R_+ \times B^2$ (briefly, $f \in C^{1,2,2}$). Let $f, \bar{f}: R_+ \times \Omega \to \mathscr{L}_2(H)$, $g, \bar{g}: R_+ \times \Omega \to H$ (here $(\Omega, \mathscr{F}, P)$ is the probability space on which the Wiener process (M_u) is defined) such that*

(a) *$f, \bar{f}, g, \bar{g}$ are measurable in both arguments,*

(b) *for each $t, f(t, \cdot)$ is $\mathscr{F}_t$-measurable.*

Then if X, Y are solutions of the equations

$$X_t = X_0 + \int_0^t f(s)\, dM_s + \int_0^t g(s)\, ds$$

$$\bar{X}_t = \bar{X}_0 + \int_0^t \bar{f}(s)\, dM_s + \int_0^t \bar{g}(s)\, ds$$

the following equality holds a.s.:

$$\begin{aligned} f(t, X_t, \bar{X}_t) = {} & f(0, X_0, \bar{X}_0) + \int_0^t \langle f^*(s) f_x(s, X_s, \bar{X}_s) \\ & + \bar{f}^*(s) f_y(s, X_s, \bar{X}_s), dM_s \rangle \\ & + \int_0^t \{ f_s(s, X_s, \bar{X}_s) + \langle g(s), f_x(s, X_s, \bar{X}_s) \rangle \\ & + \langle \bar{g}(s), f_y(s, X_s, \bar{X}_s) \rangle \\ & + \tfrac{1}{2} \operatorname{Trace}[f^*(s) f_{xx}(s, X_s, \bar{X}_s) f(s)] \\ & + \operatorname{Trace}[f^*(s) f_{xy}(s, X_s, \bar{X}_s) \bar{f}(s)] \\ & + \tfrac{1}{2} \operatorname{Trace}[\bar{f}^*(s) f_{yy}(s, X_s, \bar{X}_s) \bar{f}(s)] \}\, ds, \qquad t \geqslant 0. \end{aligned}$$

The details of the proof of this theorem follow by an argument similar to that given in the proof of Theorem 5.2 from Kuo [10, p. 155].

THEOREM 3.3. *Suppose we are given $K_1: R_+ \times B \to \mathscr{L}_2(H)$, $K_2: R_+ \times B \to H$; $L: R_+ \times B^2 \to H$, $W: R_+ \times B \to B$ which are measurable in all arguments and locally bounded.*

Assume there exists $V: R_+ \times B^2 \to R_+$, $\omega: R_+^2 \to R_+$ *and a sequence* $V_n: R_+ \times B^2 \to R_+$ *with the following properties:*

(1) $V(t, x, y) = 0$ *if and only if* $x = y$,

(2) $V_n \in C^{1,2,2}$ *and* $V_n(t, x, y) = 0$ *if* $x = y$,

(3) V_n *converges punctually to* V,

(4) ω *is continuous in both arguments,* ω *is nondecreasing and concave in the second variable and* $\omega(s, 0) = 0$,

(5) $u = 0$ *is the unique continuous solution with* $u(0) = 0$ *of the scalar equation*

$$u(t) = \int_0^t \omega(s, u(s))\, ds,$$

(6) *there exists* $f_n \in L^1_{\text{loc}}(R_+)$ *such that* f_n *converges to* 0 *in* $L^1_{\text{loc}}(R_+)$,

(7) *for every* $t \geqslant 0, f, g \in C(R_+, B)$ *we have*

$$\bar{L}(t, f, g, V_n) \leqslant \omega(t, V(t, f(t), g(t))) + f_n(t)$$

$$\frac{\partial V_n}{\partial t} \leqslant \omega(t, V) + f_n(t)$$

where for $A \in C^{1,2,2}$ *we denoted*

$$\begin{aligned} L(t, f, g, A) = {} & \left\langle K_2(t, f(t)) + L\left(t, f(t), \int_0^t W(s, f(s))\, ds\right), A_x(s, f(s), g(s))\right\rangle \\ & + \left\langle K_2(t, g(t)) + L\left(t, g(t), \int_0^t W(s, g(s))\, ds\right), A_y(s, f(s), g(s))\right\rangle \\ & + A_t(t, f(t), g(t)) \\ & + \tfrac{1}{2} \operatorname{Trace}[K_1^*(t, f(t))\, A_{xx}(t, f(t), g(t))\, K_1(t, f(t))] \\ & + \operatorname{Trace}[K_1^*(t, f(t))\, A_{xy}(t, f(t), g(t))\, K_1(t, g(t))] \\ & + \tfrac{1}{2} \operatorname{Trace}[K_1^*(t, g(t))\, A_{yy}(t, f(t), g(t))\, K_1(t, g(t))]. \end{aligned}$$

Then the solution of the stochastic integral equation

$$\begin{aligned} X_t = X_0 + \int_0^t K_1(s, X_s)\, dM_s + \int_0^t K_2(s, X_s)\, ds \\ + \int_0^t L\left(s, X_s, \int_0^s W(u, X_u)\, du\right) ds \end{aligned} \tag{3.3}$$

is pathwise unique.

Proof. Let (X_t), (Y_t) be two solutions of (3.3) with $X_0 = Y_0$ a.s. By passing to stopped processes we may assume that K_i, L, W, V are bounded. By applying Ito's lemma to $V_n(t, X_t, Y_t)$ and after that if we take the expectation we get

$$E[V_n(t, X_t, Y_t)] = E\left[\int_0^t \bar{L}(s, X, Y, V_n)\, ds\right]$$
$$\leqslant E\left[\int_0^t \omega(s, V(s, X_s, Y_s))\, ds\right] + \int_0^t f_n(s)\, ds.$$

Now by Fubini's theorem and Jensen's inequality, using the concavity of ω we obtain

$$E[V_n(t, X_t, Y_t)] \leqslant \int_0^t \omega(s, E[V(s, X_s, Y_s)])\, ds + \int_0^t f_n(s)\, ds$$

where from applying Fatou's lemma we get

$$E[V(t, X_t, Y_t)] \leqslant \int_0^t \omega(s, E[V(s, X_s, Y_s)])\, ds$$

and since $u(t) = E[V(t, X_t, Y_t)]$ is continuous, $u(0) = 0$, assumption (5) gives $u = 0$, hence $V(t, X_t, Y_t) = 0$ a.s. so that $X_t = Y_t$ a.s. for every t. The theorem is now proved.

Remark. Theorem 3.3 generalises the result of Gard [3]. Conditions which ensure the validity of assumption (5) from the above theorem can be found in Gard [3, Lemma 1] and Chen [1, Lemma 3.2].

4. Existence and Continuous Dependence

THEOREM 4.1. *Let K_i, L_i, W_i satisfy the assumption* (A) *and* L_i, W_i *satisfy the assumption* (B) *and suppose that the B-valued process φ is continuous. Moreover assume that there exists $\omega_i\colon R_+^d \times R_+ \to R_+$, $i = 1, 2, 3$, continuous in all variables and nondecreasing, uniformly Lipschitz, concave in the second variable, such that*

$$\|K_1(u, f)\|_2^2 + \|K_2(u, f)\|^2 \leqslant \omega_1(u, \|f(u)\|^2) + \omega_2\left(u, \int_0^u \omega_3(v, \|f(v)\|^2)\, d\alpha(v)\right). \quad (4.1)$$

Then Eq. (1.1) *has a solution* (X_u) *and this solution is unique. Moreover if*

$$E(\sup_{u\leqslant z}\|\varphi_u\|^2)=C(z)<\infty \tag{4.2}$$

then

$$E(\sup_{u\leqslant z}\|X_u\|^2)=\bar{C}(z)<\infty. \tag{4.3}$$

Proof. First we prove that (4.2) implies (4.3). Define $F(z)=E(\varphi_z^N\|X_z\|^2)$, where φ_z^N is the process

$$\varphi_z^N=\chi_{\{\sup_{u\leqslant z}\|X_u\|^2\leqslant N\}}.$$

Without loss of generality we may suppose that $\alpha=\beta$ and $c(z)=d(z)$ in assumptions (A) and (B) and $C(z)>0$ for every z.

As in the proof of Theorem 3.1 we obtain for $z\leqslant z_0$

$$\begin{aligned}F(z)\leqslant C(z_0)+C_1(z_0)\Big\{&\int_0^z\omega_1(u,F(u))\,du\\&+\int_0^z\omega_2\left(u,\int_0^u\omega_3(v,F(v))\,d\alpha(v)\right)du\\&+\int_0^z F(u)\,du+\int_0^z\int_0^u F(v)\,dv\,du+\int_0^z\int_0^u F(v)\,d\alpha(v)\,du\\&+\int_0^z\int_0^u\int_0^v F(w)\,d\alpha(w)\,du+\int_0^z\int_0^u\int_0^v\int_0^w F(p)\,d\alpha(p)\,dw\,d\alpha(v)\,du\Big\}.\end{aligned}$$

Now if we put $G(z)=\sup_{u\leqslant z}F(u)$ then the above inequality yields

$$G(z)\leqslant C(z_0)+C_2(z_0)\left\{\int_0^z G(u)\,du+\int_0^z\omega(u,G(u))\,du\right\} \tag{4.4}$$

where $\omega(z,x)=\omega_1(z,x)+\omega_2(z,\int_0^z\omega_3(u,x)\,d\alpha(u))$. Define

$$H(z)=C(z_0)+C_2(z_0)\int_0^z\omega(u,G(u))\,du.$$

From (4.4) we deduce

$$\frac{G(z)}{H(z)}\leqslant 1+C_2(z_0)\int_0^z\frac{G(u)}{H(u)}\,du$$

where from by using the Bellman–Gronwall inequality we obtain

$$G(z) \leqslant H(z)\left[1 + C_2(z_0)\int_0^z \exp\{C_2(z_0)(z-u)\}\, du\right] \leqslant C_3(z_0)\, H(z)$$

hence

$$G(z) \leqslant C_4(z_0)\left\{1 + \int_0^z \omega(u, G(u))\, du\right\}. \tag{4.5}$$

Relation (4.5) implies

$$G(z) \leqslant C_4(z_0)[1 + r(z)]$$

where $r(z)$ is the unique solution with $r(0) = 0$ of the equation

$$r(z) = \int_0^z \omega(u, C_4(z_0)(1 + r(u)))\, du. \tag{4.6}$$

An application of Fatou's lemma to (4.6) shows that

$$\sup_{u \leqslant z} E(\|X_u\|^2) = C_5(z) < \infty. \tag{4.7}$$

Now if in Eq. (1.1) we take $\sup_{u \leqslant z} \|X_u\|^2$ and then expectation we get, through an application of (4.7) and of Doob's and Schwartz's inequalities, the validity of (4.3).

In order to prove the existence of solution of (1.1), by technique of optional stopping, i.e., by replacing in (1.1) the process $(\varphi_{t_1,\ldots,t_d})$ with the process $(\varphi_{T\wedge t_1,\ldots,T\wedge t_d})$, where T is the $\mathscr{F}_{t,\ldots,t}$-stopping time defined by $T = \inf\{t \geqslant 0;\ \max_{t_i \leqslant t} \|\varphi_{t_1,\ldots,t_d}\| \geqslant N\}$, we may assume that (4.2) holds.

Let $L_c^2(\mathscr{F}_z)$ be the collection of d-parameter processes X which are $\mathscr{F}_z$-adapted, have continuous paths and for each z satisfy (4.3).

It is not difficult to check that for $X \in L_c^2(\mathscr{F}_z)$ the process τX given by the right member of (1.1) is also in $L_c^2(\mathscr{F}_z)$.

By using the Doob and the Schwartz inequalities we deduce that

$$E(\sup_{u \leqslant z} \|\tau X - \tau Y\|^2) \leqslant C_6(z_0) \int_0^z \sup_{v \leqslant u} E(\|X_v - Y_v\|^2)\, dv \tag{4.8}$$

for every $z \leqslant z_0$, $X, Y \in L_c^2$.

Define a sequence of processes from L_c^2 by

$$X^0 = \varphi; \qquad X^n = \tau X^{n-1} \qquad \text{for} \quad n = 1, 2, \ldots .$$

An iterated application of (4.8) implies existence of a constant $C_7(z)$ such that for every n we have

$$E(\sup_{u\leqslant z} \|X_u^{n+1} - X_u^n\|^2)$$
$$\leqslant C_7^n(z) \int_0^z \int_0^{z_1} \cdots \int_0^{z_{n-1}} \sup_{u\leqslant z_n} E(\|X_u^1 - X_u^0\|^2)\, dz_n \cdots dz_2\, dz_1$$

and since $\sup_{u\leqslant z_n} E(\|X_u^1 - X_u^0\|^2) \leqslant C_8(z)$ the above inequality gives

$$E(\sup_{u\leqslant z} \|X_u^{n+1} - X_u^n\|^2) \leqslant C_8(z)\, C_7^n(z)\, \lambda^n([0, z])/(n!)^2.$$

Therefore there exists a process X from L_c^2 such that

$$\lim_{n\to\infty} E(\sup_{u\leqslant z} \|X_u^n - X_u\|^2) = 0; \qquad \lim_{n\to\infty} \sup_{u\leqslant z} (X_u^n - X_u) = 0 \qquad \text{a.s.}$$

for each z.

It is easy to see that X is a solution of (1.1).

The uniqueness of solution to (1.1) was proved in Theorem 3.1.

THEOREM 4.2. *Let* (X^n), $n = 0, 1, \ldots$, *be solutions of*

$$X_z^n = \varphi_z^n + \int_0^z K_1^n(u, X^n)\, dM_u + \int_0^z L_1^n \left(u, X^n, \int_0^{\cdot} W_1^n(u, v, X^n)\, dM_v\right) dM_u$$
$$+ \int_0^z K_2^n(u, X^n)\, du + \int_0^z L_2^n \left(u, X^n, \int_0^{\cdot} W_2^n(u, v, X^n)\, dv\right) du$$

where the coefficients K_i^n, L_i^n, W_i^n *satisfy the conditions of Theorem* 4.1 *with the same constants* $c(z)$ *and* $d(z)$ *in the assumptions* (A) *and* (B) *and with the same functions* ω_i.

Moreover suppose that for each $N > 0$, $z > 0$

(a) $\sup_n E(\sup_{u\leqslant z} \|\varphi_u^n\|^2) = C(z) < \infty$; $\lim_{n\to\infty} E(\sup_{u\leqslant z} \|\varphi_u^n - \varphi_u^0\|^2) = 0$;

(b) $\lim_{n\to\infty} K_i^n = K_i^0$; $\lim_{n\to\infty} L_i^n = L_i^0$; $\lim_{n\to\infty} W_i^n = W_i^0$ *punctually for* $i = 1, 2$.

Then

$$\lim_{n\to\infty} E(\sup_{u\leqslant z} \|X_0^n - X_u^0\|^2) = 0.$$

Proof. We have

$$\sup_{u\leqslant z}\|X_u^n - X_u^0\|^2 \leqslant 8\left\{\sup_{u\leqslant z}\|\varphi_u^n - \varphi_u^0\|^2 + \sup_{u\leqslant z}\|(\tau_n X^n - \tau_n X^0)(u)\|^2 + \sum_{i=1}^{6} J_i(z)\right\} \tag{4.9}$$

where τ_n is the operator τ defined as in the proof of Theorem 4.1 with K_i^n, L_i^n, W_i^n and

$$J_1^n(z) = \sup_{u\leqslant z}\left\|\int_0^u [K_1^n(v, X^0) - K_1^0(v, X^0)]\, dM_u\right\|^2$$

$$J_2^n(z) = \sup_{u\leqslant z}\left\|\int_0^u [K_2^n(v, X^0) - K_2^0(v, X^0)]\, du\right\|^2$$

$$J_3^n(z) = \sup_{u\leqslant z}\left\|\int_0^u \left[L_1^n\left(v, X^0, \int_0^{\cdot} W_1^n(v, p, X^0)\, dM_p\right) - L_1^n\left(v, X^0, \int_0^{\cdot} W_1^0(v, p, X^0)\, dM_p\right)\right] dM_v\right\|^2$$

$$J_4^n = \sup_{u\leqslant z}\left\|\int_0^u \left[L_1^n\left(v, X^0, \int_0^{\cdot} W_1^0(v, p, X^0)\, dM_p\right) - L_1^0\left(v, X^0, \int_0^{\cdot} W_1^0(v, p, X^0)\, dM_p\right)\right] dM_v\right\|^2$$

$$J_5^n(z) = \sup_{u\leqslant z}\left\|\int_0^u \left[L_2^n\left(v, X^0, \int_0^{\cdot} W_2^n(v, p, X^0)\, dp\right) - L_2^n\left(v, X^0, \int_0^{\cdot} W_2^0(v, p, X^0)\, dp\right)\right] dv\right\|^2$$

$$J_6^n(z) = \sup_{u\leqslant z}\left\|\int_0^u \left[L_2^n\left(v, X^0, \int_0^{\cdot} W_2^0(v, p, X^0)\, dp\right) - L_2^0\left(v, X^0, \int_0^{\cdot} W_2^0(v, p, X^0)\, dp\right)\right] dv\right\|^2.$$

Using assumption (A) and Doob's and Schwartz's inequalities we get for $z \leqslant z_0$

$$E(\sup_{u\leqslant z}\|(\tau_n X^n - \tau_n X^0)(u)\|^2) + E(J_3^n(z)) + E(J_5^n(z)) \leqslant C(d, z_0)\int_0^z E(\sup_{v\leqslant u}\|X_v^n - X_v^0\|^2)\, du$$

(here $C(d, z_0)$ is a constant depending only on d, z_0). Also, using assumption (B), Doob's and Schwartz's inequalities, (4.1), (4.3) and the dominated convergence theorem, we obtain that

$$E(J_2^n(z)) + E(J_4^n(z)) + E(J_6^n(z)) \to 0.$$

Hence by taking expectation in (4.9) we get for $\varepsilon > 0$ and n large enough

$$E(\sup_{u \leqslant z} \|X_u^n - X_u^0\|^2) \leqslant 8 \left\{ \varepsilon + C(d, z_0) \int_0^z E(\sup_{v \leqslant u} \|X_v^n - X_v^0\|^2)\, du \right\}$$

where from by applying the Bellman–Gronwall inequality we deduce

$$E(\sup_{u \leqslant z} \|X_u^n - X_u^0\|^2) \leqslant 8\varepsilon \left[1 + 8C(d, z_0) \int_0^z \exp\{8C(d, z_0)(z-u)\}\, du \right]$$

and this completes the proof of the theorem.

Remark. Theorem 4.1 generalises the results of Protter [14, Theorem 3.1] and of Yeh [19, Theorem 3.12].

Versions of the convergence theorem 4.2 in the case of finite-dimensional state space can be found in Protter [14] and Tudor [17].

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